

The Disc Amplitude of the Dijkgraaf-Vafa Theory: $1/N$ Expansion vs Complex Curve Analysis

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ABSTRACT: According to Dijkgraaf and Vafa the effective glueball superpotential of the $\mathcal{N} = 1$ supersymmetric QCD coupled with an adjoint chiral multiplet is given by the planar amplitude in the $\frac{1}{N}$ expansion of a matrix model. It was shown that, when the $\mathcal{N} = 1$ supersymmetric QCD is coupled with fundamental chiral multiplets as well, the effective glueball superpotential is modified by the disc amplitude of the generalized matrix model. The diagrammatic computation of this disc amplitude is fairly involved for the multi-cut solution. Instead we compute it with recourse to the complex analysis of the hyperelliptic curve. The result is given in series of the gluino condensation S_i . The explicit computation for the generic multi-cut solution is done up to order S^3 . It is systematic so that it can be extended to higher orders.

KEYWORDS: Matrix Models, Supersymmetric Effective Theories, Nonperturbative Effects, $1/N$ Expansion.

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1. Introduction

Dijkgraaf and Vafa discovered a close relation between the $\mathcal{N} = 1$ supersymmetric QCD and a matrix model. They claimed that the glueball superpotential of the $\mathcal{N} = 1$ supersymmetric QCD coupled with an adjoint chiral multiplet can be computed by the $\frac{1}{N}$ expansion of an associated matrix model[1]. It was confirmed by subsequent works by many people. The effective theory of the $\mathcal{N} = 1$ supersymmetric QCD given by such a glueball superpotential is called the Dijkgraaf-Vafa (DV) theory. In [1] they developed a remarkable technique to compute the planar diagram amplitude of the matrix model. Namely the computation, which was normally done by the perturbative sum of the planar diagrams, was equivalently replaced by the analysis of the hyperelliptic curve on the Riemann surface. This revealed important aspects underlying in the $\mathcal{N} = 1$ supersymmetric QCD like the duality[2], the integrable hierarchy, *etc*, and attracted a renewed attention[3, 4] to the Seiberg-Witten theory for the $N = 2$ supersymmetric QCD[5].

In [6, 7, 8, 9] the relation between the $\mathcal{N} = 1$ supersymmetric QCD and a matrix model was generalized for the case in which the $\mathcal{N} = 1$ supersymmetric QCD is coupled with fundamental chiral multiplets as well. The presence of fundamental chiral multiplets modifies the glueball superpotential of the $\mathcal{N} = 1$ supersymmetric QCD. It was shown that the modification can be captured by computing the planar diagrams with boundary (discs) of the matrix model which is generalized accordingly. The disc amplitude is the perturbative sum of such disc diagrams. It was computed to all orders of the gluino condensation S for the 1-cut solution[6], but merely to

order S^2 for the n -cut solution[8]. The computation to higher orders for the n -cut solution is extremely involved. In this paper we will discuss that the disc amplitude for the n -cut solution may be computed to higher orders in a rather systematic way, if one has recourse to the complex analysis of the hyperelliptic curve. Indeed the planar amplitude for the 2-cut solution was calculated along this line to order S^3 in [10]. This calculation to order S^3 was generalized for the n -cut solution in [11] later on. Our aim of this paper is to generalize the calculation of the planar amplitude in [11] to that of the disc amplitude.

The paper is organized as follows. We give a short review on the DV theory coupled with fundamental chiral multiplets in section 2. The disc amplitude of the associated matrix model is given in terms of the (hyper)elliptic function. As a warming-up it is evaluated for the 1-cut solution in section 3. The evaluation is exact for this case. For the multi-cut solution it can be done only in series of the gluino condensation S . In section 4 the disc amplitude for the 2-cut solution is evaluated to order S^3 . In section 5 this evaluation is generalized for the n -cut solution.

2. A Short Review on the DV theory

We consider the $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory with the superpotential

$$W(\Phi, Q^I, \tilde{Q}_I) = \text{tr} V(\Phi) + \sum_{I=1}^{N_f} (m_I \tilde{Q}_I Q^I + Q_I \Phi Q^I), \quad (2.1)$$

Here Φ is a adjoint chiral multiplet and Q^I, \tilde{Q}_I are N_f chiral multiplets in the fundamental representation. $V(\Phi)$ is a polynomial scalar potential of degree M_0 given by

$$V'(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{M_0}). \quad (2.2)$$

When the $U(N)$ gauge group symmetry is broken to $\Pi_i U(N_i)$ with $\sum_{i=1}^n N_i = N$, this gauge theory is described by the effective action

$$W_{eff} = \sum_{i=1}^n N_i \frac{\partial F_2(S)}{\partial S_i} + F_1(S) + 2\pi i \tau_0 \sum_{i=1}^n S_i, \quad (2.3)$$

for the gluino condensation S_i . It was claimed that $F_2(S)$ and $F_1(S)$ are the planar and disc amplitudes in the $\frac{1}{M}$ expansion of the partition function of the associated matrix model

$$\begin{aligned} Z &= \int d\phi dq^I d\tilde{q}_I \exp \left[-\frac{1}{g_s} W(\phi, q^I, \tilde{q}_I) \right] \\ &= \exp \left[-\sum_{\chi \leq 2} g_s^{-\chi} F_\chi(S) \right]. \end{aligned}$$

Here ϕ is a $M \times M$ matrix and $q^I(\tilde{q}^I)$ are M dimensional vectors. W is the super-potential given by (2.1) so that the matrix model is gauge-invariant under $U(M)$. The gluino condensation S_i is identified with the 'tHooft coupling $g_s M_i$, when the gauge symmetry $U(M)$ is broken to $\Pi_i U(M_i)$ with $\sum_{i=1}^n M_i = M$. The planar and disc amplitudes, respectively given by F_2 and F_1 , are calculated by an alternative method. Namely we think of the bosonic matrix model

$$Z = \int d\phi \exp\left[-\frac{1}{g_s} \text{tr} V(\phi)\right].$$

Then we obtain the saddle point equation for the eigenvalues distributed with density $\varrho(\lambda)$

$$y \equiv V'(x) + 2 \int d\lambda \frac{\varrho(\lambda)}{x - \lambda} = 0. \quad (2.4)$$

Here y is the quantum deformed force of the matrix model. For the n -cut solution it may be written as a hyperelliptic function such that

$$y = (x - \mu_1)(x - \mu_1) \cdots (x - \mu_{M_0-n}) \sqrt{(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_{2n})}. \quad (2.5)$$

Comparing the polynomial part in the expansion of (2.4) and (2.5) at $x = \infty$ gives the zero-points $\alpha_a, a = 1, 2, \dots, M_0$ as functions of μ 's and λ 's. We consider the DV differential ydx defined with such a hyperelliptic function. For the n -cut solution the gluino condensation is given by

$$S_i = \frac{1}{4\pi i} \int_{A_i} ydx. \quad (2.6)$$

The planar amplitude[1] was given by

$$\frac{\partial F_2}{\partial S_i} = \int_{\alpha_i}^{\Lambda} ydx, \quad (2.7)$$

while the disc amplitude[8, 9] by

$$F_1 = \frac{1}{2} \sum_{I=1}^{N_f} \left[\int_{-m_I}^{\Lambda} ydx - V(\Lambda) + V(-m_I) \right], \quad (2.8)$$

with a cut-off parameter Λ .

We want to calculate this disc amplitude for the n -cut solution and compare it with the result obtained by the $\frac{1}{M}$ expansion. For simplicity we shall do it simply assuming that $M_0 = n$.

3. The 1-cut Solution

As a warming-up we start with the simplest example, *i.e.*, the matrix model with the scalar potential

$$V(x) = \frac{1}{2}\xi x^2,$$

which is coupled with one chiral matter multiplet. We consider the DV differential

$$ydx = \sqrt{V'(x)^2 - 4\xi S}dx,$$

on the Riemann surface with one branch-cut. Note that the branch points are chosen so that (2.6) is satisfied, *i.e.*,

$$S = \frac{1}{2\pi i} \int_{\sqrt{-\frac{4S}{\xi}}}^{\sqrt{\frac{4S}{\xi}}} ydx.$$

The disc amplitude (2.8) is exactly calculated as

$$F_1(S) = -\frac{S}{2} - \frac{1}{4\alpha}(\sqrt{1-4\alpha S} - 1) + S \log \frac{1 + \sqrt{1-4\alpha S}}{2} + S \log \frac{m}{\Lambda},$$

with $\alpha = \frac{1}{m^2\xi}$. This is in complete agreement with the result obtained by the $\frac{1}{N}$ expansion[6]. The log-term which is linear in S is a non-perturbative part and may be absorbed by the last term in the effective action (2.3). A similar calculation for the 1-cut solution was done for a variant of the matrix model in [12].

4. The 2-cut Solution

Next we study the matrix model in which the scalar potential is given by

$$V'(x) = (x - \alpha_1)(x - \alpha_2). \quad (4.1)$$

In [8] they calculated Feynmann diagrams depicted in Fig. 1 to find the disc amplitude to order S^2 :

$$F_1(S) = - \sum_{I=1}^{N_f} \sum_{i=1}^2 \left[\frac{S_i^2}{\Delta_i(\alpha) f_{iI}} \frac{1}{\alpha_{ij}} - 2 \sum_{j \neq i}^2 \frac{S_i S_j}{\Delta_i(\alpha) \alpha_{ij} f_{iI}} + \frac{1}{2} \frac{S_i^2}{\Delta_i(\alpha) f_{iI}^2} \right]. \quad (4.2)$$

Here $\Delta_i(\alpha) = \prod_{j \neq i}^2 \alpha_{ij}$, $f_{iI} = m_I + \alpha_i$, $\alpha_{ij} = \alpha_i - \alpha_j$. In this section we shall evaluate the formula (2.8) for the 2-cut solution to order S^3 and show that it indeed reproduces the result (4.2) to order S^2 . To this end we consider the DV differential

$$ydx = \sqrt{\left((x - \gamma_1)^2 - \rho_1^2\right)\left((x - \gamma_2)^2 - \rho_2^2\right)}dx, \quad (4.3)$$

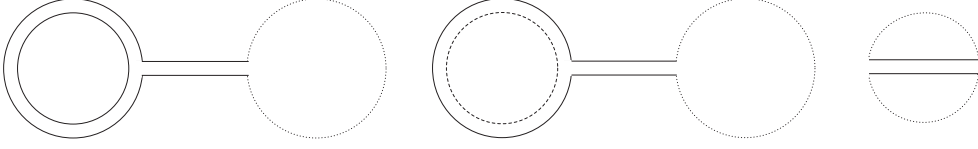


Figure 1: The Feynman diagrams to order S^2 corresponding to the three terms in (4.2). The solid double line and the single dashed line are propagators for ϕ and q^I respectively. The solid and dashed double line is a propagator for ghosts.

on the Riemann surface with two branch-cuts $[\gamma_1 - \alpha_1, \gamma_1 + \alpha_1]$ and $[\gamma_2 - \alpha_2, \gamma_2 + \alpha_2]$. Here y may be written in the form (2.4). Hence comparing the polynomial part in the expansion of (2.4) and (4.3) at $x = \infty$ yields the relations $\alpha_i(\rho^2, \gamma)$:

$$\alpha_1 + \alpha_2 = \gamma_1 + \gamma_2, \quad (4.4)$$

$$(\alpha_1 + \alpha_2)^2 + 2\alpha_1\alpha_2 = \gamma_1^2 + \gamma_2^2 + 4\gamma_1\gamma_2 - \rho_1^2 - \rho_2^2. \quad (4.5)$$

With the DV differential (4.3) we will calculate the disc amplitude (2.8). A suitable technique to this end was developed for the calculation of the planar amplitude in [11]. The calculation goes following the steps:

1. evaluation of the period integrals

$$S_i = \frac{1}{2\pi i} \int_{\gamma_i - \rho_i}^{\gamma_i + \rho_i} y dx, \quad (4.6)$$

which provides explicit formulae $S_i(\rho^2, \gamma)$ for gluino condensation.

2. inversion of the formula $S_i(\rho^2, \gamma)$ which provides $\rho_i^2(S, \gamma)$ in series of S_i with coefficients made out of γ_i .
3. inversion of the formulae $\alpha_i(\rho^2, \gamma)$, given by (4.4) and (4.5), which provides $\gamma_i(\rho^2, \alpha)$ in series of ρ_i^2 with coefficients made out of α_i .
4. evaluation of the DV differential (4.3) in series of ρ_i^2 with coefficients made out of γ_i^2
5. expansion of ρ_i^2 and γ_i in the above result in series of S_i with the coefficients made out of α_i by iterating the inversion formulae obtained at Step 2 and 3.

Step 1

After the shift $x' = x - \gamma_1$ we expand the DV differential (4.3) in series of ρ_2^2 as

$$y dx = \sum_{m=0}^{\infty} c_m \rho_2^{2m} (x' + \gamma_{12})^{1-2m} \sqrt{x'^2 - \rho_1^2} dx', \quad (4.7)$$

where $\gamma_{12} = \gamma_1 - \gamma_2$ and $c_m = \frac{\Gamma(m-\frac{1}{2})}{m!\Gamma(-\frac{1}{2})}$. Write $(x' + \gamma_{12})^{1-2m}$ for $m \geq 1$ as $(2m-2)$ -ple derivative of $(x' + \gamma_{12})^{-1}$ by γ_{12} . Then it takes the form

$$ydx = \left[(x' + \gamma_{12})\sqrt{x'^2 - \rho_1^2} + \sum_{m \geq 1} \frac{c_m}{(2m-2)!} \rho_2^{2m} \left(\frac{\partial}{\partial \gamma_{12}} \right)^{2m-2} \frac{\sqrt{x'^2 - \rho_1^2}}{x' + \gamma_{12}} \right] dx'. \quad (4.8)$$

We integrate the DV differential of this form along the A_1 -cycle surrounding the branch-cuts $[\gamma_1 - \rho_1, \gamma_1 + \rho_1]$. We find

$$i \int_{-\rho_1}^{\rho_1} dx' (x' + \gamma_{12}) \sqrt{\rho_1^2 - x'^2} = \frac{i\pi}{2} \gamma_{12} \rho_1^2, \quad (4.9)$$

for the first piece and

$$i \int_{-\rho_1}^{\rho_1} dx' \frac{\sqrt{\rho_1^2 - x'^2}}{x' + \gamma_{12}} = i\pi (-\sqrt{\gamma_{12}^2 - \rho_1^2} + \gamma_{12}), \quad (4.10)$$

for the second piece. Putting them together in (4.6) then we get the expression for the gluino condensation S_1

$$S_1 = \frac{1}{4} \gamma_{12} \rho_1^2 - \frac{\rho_1^2 \rho_2^2}{8\gamma_{12}} - \frac{\rho_1^2 \rho_2^2 (\rho_1^2 + \rho_2^2)}{32\gamma_{12}^3} + O(\rho^8). \quad (4.11)$$

Similarly we get for the gluino condensation S_2

$$S_2 = -\frac{1}{4} \gamma_{12} \rho_2^2 + \frac{\rho_1^2 \rho_2^2}{8\gamma_{12}} + \frac{\rho_1^2 \rho_2^2 (\rho_1^2 + \rho_2^2)}{32\gamma_{12}^3} + O(\rho^8). \quad (4.12)$$

Step 2

Inverting the relations (4.11) and (4.12) we obtain

$$\rho_1^2 = \frac{4S_1}{\gamma_{12}} - \frac{8S_1 S_2}{\gamma_{12}^4} - \frac{24S_1 S_2 (S_1 - S_2)}{\gamma_{12}^7} + O(S^4), \quad (4.13)$$

$$\rho_2^2 = -\frac{4S_2}{\gamma_{12}} - \frac{8S_1 S_2}{\gamma_{12}^4} - \frac{24S_1 S_2 (S_1 - S_2)}{\gamma_{12}^7} + O(S^4). \quad (4.14)$$

Step 3

Inverting the relations (4.4) and (4.5) we obtain

$$\begin{aligned} \gamma_1 &= \frac{1}{2}(\alpha_1 + \alpha_2) + \frac{1}{2}\sqrt{(\alpha_1 - \alpha_2)^2 - 2(\rho_1^2 + \rho_2^2)}, \\ &= \alpha_1 - \frac{\rho_1^2 + \rho_2^2}{2\alpha_{12}} - \frac{(\rho_1^2 + \rho_2^2)^2}{4\alpha_{12}^3} + O(\rho^6), \end{aligned} \quad (4.15)$$

$$\begin{aligned} \gamma_2 &= \frac{1}{2}(\alpha_1 + \alpha_2) - \frac{1}{2}\sqrt{(\alpha_1 - \alpha_2)^2 - 2(\rho_1^2 + \rho_2^2)} \\ &= \alpha_2 + \frac{\rho_1^2 + \rho_2^2}{2\alpha_{12}} + \frac{(\rho_1^2 + \rho_2^2)^2}{4\alpha_{12}^3} + O(\rho^6). \end{aligned} \quad (4.16)$$

Step 4

We expand the DV differential (4.8) in series of ρ_1^2 as well.

$$\begin{aligned}
ydx = & \left[V'(x) - \frac{\gamma_{12}}{2} \left(\frac{\rho_1^2}{x'} - \frac{\rho_2^2}{x' + \gamma_{12}} \right) - \frac{1}{8} \left\{ \frac{\rho_1^4}{x'^2} + \frac{\rho_2^4}{(x' + \gamma_{12})^2} \right\} \right. \\
& - \frac{\gamma_{12}}{8} \left\{ \frac{\rho_1^4}{x'^3} - \frac{\rho_2^4}{(x' + \gamma_{12})^3} \right\} + \frac{\rho_1^2 \rho_2^2}{4\gamma_{12}} \left(\frac{1}{x'} - \frac{1}{x' + \gamma_{12}} \right) \\
& - \frac{1}{16} \left\{ \frac{\rho_1^6}{x'^4} + \frac{\rho_2^6}{(x' + \gamma_{12})^4} \right\} - \frac{\gamma_{12}}{16} \left\{ \frac{\rho_1^6}{x'^5} - \frac{\rho_2^6}{(x' + \gamma_{12})^5} \right\} \\
& + \frac{1}{16\gamma_{12}} \left\{ \frac{\rho_1^4 \rho_2^2}{x'^3} - \frac{\rho_1^2 \rho_2^4}{(x' + \gamma_{12})^3} \right\} - \frac{1}{16\gamma_{12}^2} \left\{ \frac{\rho_1^4 \rho_2^2}{x'^2} + \frac{\rho_1^2 \rho_2^4}{(x' + \gamma_{12})^2} \right\} \\
& \left. + \frac{\rho_1^2 \rho_2^2 (\rho_1^2 + \rho_2^2)}{16\gamma_{12}^3} \left(\frac{1}{x'} - \frac{1}{x' + \gamma_{12}} \right) + O(\rho^8) \right] dx'.
\end{aligned} \tag{4.17}$$

Here $V'(x)$ is given by

$$V'(x) = x'^2 + \gamma_{12}x' - \frac{1}{2}(\rho_1^2 + \rho_2^2),$$

which is nothing but the potential (4.1) owing to (4.4) and (4.5). After shifting the variable x' back to x we write (4.17) expressing ρ_i^2 and γ_i in terms S_i and α_i with recourse to (4.13)~(4.16). In particular the terms of order ρ^2 and ρ^4 in (4.17) read

$$\begin{aligned}
\frac{\gamma_{12}}{2} \left(\frac{\rho_1^2}{x'} - \frac{\rho_2^2}{x' + \gamma_{12}} \right) &= 2 \left\{ S_1 - \frac{2}{\alpha_{12}^3} S_1 S_2 - \frac{30}{\alpha_{12}^6} S_1 S_2 (S_1 - S_2) \right\} \frac{1}{x - \alpha_1} \\
&+ 2 \left\{ S_2 + \frac{2}{\alpha_{12}^3} S_1 S_2 + \frac{30}{\alpha_{12}^6} S_1 S_2 (S_1 - S_2) \right\} \frac{1}{x - \alpha_2} \\
&- 4 \left\{ \frac{1}{\alpha_{12}^2} S_1 (S_1 - S_2) + \frac{2}{\alpha_{12}^5} S_1 (3S_1^2 - 9S_1 S_2 + 4S_2^2) \right\} \frac{1}{(x - \alpha_1)^2} \\
&+ 4 \left\{ \frac{1}{\alpha_{12}^2} S_2 (S_1 - S_2) + \frac{2}{\alpha_{12}^5} S_2 (4S_1^2 - 9S_1 S_2 + 3S_2^2) \right\} \frac{1}{(x - \alpha_2)^2} \\
&+ \frac{8}{\alpha_{12}^4} \frac{S_1 (S_1 - S_2)^2}{(x - \alpha_1)^3} + \frac{8}{\alpha_{12}^4} \frac{S_2 (S_1 - S_2)^2}{(x - \alpha_2)^3}, \\
\frac{1}{8} \left\{ \frac{\rho_1^4}{x'^2} + \frac{\rho_2^4}{(x' + \gamma_{12})^2} \right\} &= 2 \left\{ \frac{S_1^2}{\alpha_{12}^2} + \frac{4}{\alpha_{12}^5} S_1 (2S_1^2 - 3S_1 S_2) \right\} \frac{1}{(x - \alpha_1)^2} \\
&+ 2 \left\{ \frac{S_2^2}{\alpha_{12}^2} + \frac{4}{\alpha_{12}^5} S_2 (3S_1 S_2 - 2S_2^2) \right\} \frac{1}{(x - \alpha_2)^2} \\
&- \frac{8}{\alpha_{12}^4} \frac{S_1^2 (S_1 - S_2)}{(x - \alpha_1)^3} + \frac{8}{\alpha_{12}^4} \frac{S_2 (S_1 - S_2)}{(x - \alpha_2)^3}, \\
\frac{\gamma_{12}}{8} \left\{ \frac{\rho_1^4}{x'^3} - \frac{\rho_2^4}{(x' + \gamma_{12})^3} \right\} &= 2 \left\{ \frac{S_1^2}{\alpha_{12}} + \frac{4}{\alpha_{12}^4} S_1^2 (S_1 - 2S_2) \right\} \frac{1}{(x - \alpha_1)^3} \\
&- 2 \left\{ \frac{S_2^2}{\alpha_{12}} + \frac{4}{\alpha_{12}^4} S_2^2 (2S_1 - S_2) \right\} \frac{1}{(x - \alpha_2)^3}
\end{aligned}$$

$$\begin{aligned}
& -\frac{12}{\alpha_{12}^3} \frac{S_1^2(S_1 - S_2)}{(x - \alpha_1)^4} - \frac{12}{\alpha_{12}^3} \frac{S_2^2(S_1 - S_2)}{(x - \alpha_2)^4}, \\
\frac{\rho_1^2 \rho_2^2}{4\gamma_{12}} \left(\frac{1}{x'} - \frac{1}{x' + \gamma_{12}} \right) &= -4 \left\{ \frac{S_1 S_2}{\alpha_{12}^3} + \frac{14}{\alpha_{12}^6} S_1 S_2 (S_1 - S_2) \right\} \left(\frac{1}{x - \alpha_1} - \frac{1}{x - \alpha_2} \right) \\
&+ \frac{8}{\alpha_{12}^5} S_1 S_2 (S_1 - S_2) \left\{ \frac{1}{(x - \alpha_1)^2} + \frac{1}{(x - \alpha_2)^2} \right\}.
\end{aligned}$$

With these the disc amplitude (2.8) is evaluated as

$$\begin{aligned}
F_1(S) &= - \sum_{I=1}^{N_f} \left[-S_1 \log \frac{m_I + \alpha_1}{\Lambda} - S_2 \log \frac{m_I + \alpha_2}{\Lambda} \right. \\
&+ \left(\frac{1}{\alpha_{12}^2} (S_1^2 - 2S_1 S_2) + \frac{2}{\alpha_{12}^5} (2S_1^3 - 9S_1^2 S_2 + 6S_1 S_2^2) \right) \frac{1}{m_I + \alpha_1} \\
&+ \left(\frac{1}{\alpha_{12}^2} (S_2^2 - 2S_1 S_2) - \frac{2}{\alpha_{12}^5} (2S_2^3 - 9S_1^2 S_2 + 6S_1 S_2^2) \right) \frac{1}{m_I + \alpha_2} \\
&+ \left(\frac{S_1^2}{2\alpha_{12}} + \frac{1}{\alpha_{12}^4} (2S_1^3 - 5S_1^2 S_2 + 2S_1 S_2^2) \right) \frac{1}{(m_I + \alpha_1)^2} \\
&+ \left(-\frac{S_1^2}{2\alpha_{12}} + \frac{1}{\alpha_{12}^4} (2S_2^3 - 5S_1 S_2^2 + 2S_1^2 S_2) \right) \frac{1}{(m_I + \alpha_2)^2} \\
&+ \frac{2}{3\alpha_{12}^3} \left\{ (2S_1^3 - 3S_1^2 S_2) \frac{1}{(m_I + \alpha_1)^3} - (2S_2^3 - 3S_1 S_2^2) \frac{1}{(m_I + \alpha_2)^3} \right\} \\
&+ \left. \frac{1}{2\alpha_{12}^2} \left\{ \frac{S_1^3}{(m_I + \alpha_1)^4} + \frac{S_2^3}{(m_I + \alpha_2)^4} \right\} + O(S^4) \right]. \tag{4.18}
\end{aligned}$$

The log-terms linear in S is a non-perturbative part which may be absorbed by the last term in the effective action (2.3). The terms of order S^2 are in complete agreement with the result (4.2) by the $\frac{1}{N}$ expansion.

5. The n -cut Solution

We shall generalize the calculation for the 2-cut solution to that of the n -cut solution. That is, taking the matrix model in which the scalar potential is given by

$$V'(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n). \tag{5.1}$$

we consider the DV differential

$$y dx = \sqrt{\prod_{i=1}^n \left((x - \gamma_i)^2 - \rho_i^2 \right)} dx, \tag{5.2}$$

on the Riemann surface with n branch-cuts. Again we calculate the disc amplitude following Step 1 to Step 5.

Step 1 and Step 2

The inversion formula $\rho_i^2(S, \gamma)$ was given in the expansion form[11]

$$\rho_i^2 = \frac{4S_i}{\Delta_i(\gamma)} \left(1 - \frac{1}{2\Delta_i(\gamma)} \sum_{\substack{j \neq k \\ j, k \neq i}} \frac{S_i}{\gamma_{ij}\gamma_{ik}} + 2 \sum_{j \neq i} \frac{S_j}{\gamma_{ij}^2 \Delta_j(\gamma)} \right) + O(S^3), \quad (5.3)$$

in which

$$\Delta_i(\gamma) = \prod_{i \neq j}^n \gamma_{ij}, \quad \gamma_{ij} = \gamma_i - \gamma_j.$$

As it will be clear, the expansion up to order S^2 is sufficient for the calculation of the disc amplitude to order S^3 .

Step 3

We expand y , given by (2.4) and (5.2), at $x = \infty$. Comparing the polynomial part of both expansions yields[11]

$$2 \sum_{i=1}^n \alpha_i^l = \sum_{i=1}^n \left[(\gamma_i + \rho_i)^l + (\gamma_i - \rho_i)^l \right], \quad 1 \leq l \leq n. \quad (5.4)$$

We shall express γ_i in terms of ρ and α by solving this set of equations recursively. First of all expand (5.4) in series of ρ^2 as

$$\sum_{i=1}^n \alpha_i^l = \sum_{i=1}^n \left[\gamma_i^l + \binom{l}{2} \gamma_i^{l-2} \rho_i^2 + \binom{l}{4} \gamma_i^{l-4} \rho_i^4 + O(\rho^6) \right]. \quad (5.5)$$

Assume that γ_i is further expanded in the form

$$\gamma_i = \alpha_i + \sigma_i^{(2)} + \sigma_i^{(4)} + O(\rho^6), \quad (5.6)$$

with $\sigma_i^{(n)}$ the term of order ρ^n . Putting this in (5.5) we get the recursion relations for $\sigma_i^{(n)}$

$$\sum_{i=1}^n \left[\alpha_i^{l-1} \sigma_i^{(2)} + \frac{1}{2} (l-1) \alpha_i^{l-2} \rho_i^2 \right] = 0, \quad (5.7)$$

$$\begin{aligned} \sum_{i=1}^n \left[\alpha_i^{l-1} \sigma_i^{(4)} + \frac{1}{2} (l-1) \alpha_i^{l-2} (\sigma_i^{(2)})^2 + \frac{1}{2} (l-1)(l-2) \alpha_i^{l-3} \sigma_i^{(2)} \rho_i^2 \right. \\ \left. + \frac{1}{4!} (l-1)(l-2)(l-3) \alpha_i^{l-4} \rho_i^4 \right] = 0, \end{aligned} \quad (5.8)$$

and so on. Note that the identity

$$\frac{P^{[i]}(\alpha_j)}{\Delta_i(\alpha)} = \delta_{ij}, \quad (5.9)$$

for the quantity

$$P^{[i]}(x) = \frac{V'(x)}{x - \alpha_i} = \prod_{l \neq i}^n (x - \alpha_l). \quad (5.10)$$

Using the expansion

$$P^{[i]}(x) = \sum_{l=1}^n c_l^{[i]}(\alpha) x^{l-1},$$

(5.9) becomes

$$\frac{1}{\Delta_i(\alpha)} \sum_{l=1}^n c_l^{[i]}(\alpha) \alpha_j^{l-1} = \delta_{ij}.$$

By means of this identity the recursion relations (5.7) and (5.8) can be solved for $\sigma_i^{(n)}$ as

$$\sigma_i^{(2)} = -\frac{1}{2\Delta_i(\alpha)} \sum_{j=1}^n P^{[i]'}(\alpha_j) \rho_j^2, \quad (5.11)$$

$$\begin{aligned} \sigma_i^{(4)} = & -\frac{1}{2\Delta_i(\alpha)} \sum_{j=1}^n P^{[i]'}(\alpha_j) (\sigma_i^{(2)})^2 - \frac{1}{2\Delta_i(\alpha)} \sum_{j=1}^n P^{[i]''}(\alpha_j) \sigma_i^{(2)} \rho_j^2 \\ & - \frac{1}{24\Delta_i(\alpha)} \sum_{j=1}^n P^{[i]'''}(\alpha_j) \rho_j^4. \end{aligned} \quad (5.12)$$

Here $P^{[i]'}(x)$, $P^{[i]''}(x)$, $P^{[i]'''}(x)$ are derivatives of the quantity (5.10) with respect to x . Putting (5.11) and (5.12) in (5.6) we obtain the formula generalizing (4.15) and (4.16) for the n -cut solution.

Step 4

Expanding y given by the DV differential (5.2) in series of ρ^2 we get

$$\begin{aligned} y = & - \sum_{i=1}^n \left[\frac{1}{2} \frac{\rho_i^2}{x - \gamma_i} + \frac{1}{8} \frac{\rho_i^4}{(x - \gamma_i)^3} + \frac{1}{16} \frac{\rho_i^6}{(x - \gamma_i)^5} \right] \tilde{P}^{[i]}(x) \\ & + \sum_{i \neq j}^n \left[\frac{1}{8} \frac{\rho_i^2 \rho_j^2}{(x - \gamma_i)(x - \gamma_j)} + \frac{1}{16} \frac{\rho_i^2 \rho_j^4}{(x - \gamma_i)(x - \gamma_j)^3} \right] \tilde{P}^{[ij]}(x) \\ & - \frac{1}{48} \sum_{i \neq j \neq k}^n \frac{\rho_i^2 \rho_j^2 \rho_k^2}{(x - \gamma_i)(x - \gamma_j)(x - \gamma_k)} \tilde{P}^{[ijk]}(x) + O(\rho^8), \end{aligned} \quad (5.13)$$

with

$$\begin{aligned} \tilde{P}^{[i]}(x) &= \prod_{l \neq i}^n (x - \gamma_l), & \tilde{P}^{[ij]}(x) &= \prod_{l \neq i, j}^n (x - \gamma_l), \\ \tilde{P}^{[ijk]}(x) &= \prod_{l \neq i, j, k}^n (x - \gamma_l). \end{aligned}$$

When $\tilde{P}^{[i]}(x)$, $\tilde{P}^{[ij]}(x)$ and $\tilde{P}^{[ijk]}(x)$ are expanded at $x = \gamma_i, \gamma_j, \gamma_k$ in Taylor series, it takes the form

$$y = V'(x) + \sum_{\nu=1}^5 \sum_{i=1}^n \frac{a_{\nu i}(\rho^2, \gamma)}{(x - \gamma_i)^\nu} + O(\rho^8),$$

with

$$\begin{aligned} a_{1i}(\rho^2, \gamma) = & -\frac{\rho_i^6}{384} \tilde{P}^{[i]''''}(\gamma_i) - \frac{\rho_i^4}{16} \tilde{P}^{[i]''}(\gamma_i) - \frac{\rho_i^2}{2} \tilde{P}^{[i]}(\gamma_i) \\ & + \sum_{j \neq i}^n \left[-\frac{\rho_i^4 \rho_j^2}{16 \gamma_{ij}^2} \tilde{P}^{[ij]'}(\gamma_i) + \frac{\rho_i^4 \rho_j^2}{32 \gamma_{ij}} \tilde{P}^{[ij]''}(\gamma_i) \right. \\ & \quad \left. + \frac{\rho_i^2 \rho_j^2 (\rho_i^2 + \rho_j^2)}{16 \gamma_{ij}^3} \tilde{P}^{[ij]}(\gamma_i) + \frac{\rho_i^2 \rho_j^2}{4 \gamma_{ij}} \tilde{P}^{[ij]}(\gamma_i) \right] \\ & - \sum_{\substack{j \neq k \\ j, k \neq i}}^n \frac{\rho_i^2 \rho_j^2 \rho_k^2}{24} \left(\frac{1}{\gamma_{ij} \gamma_{ik}} + \frac{1}{\gamma_{ij} \gamma_{jk}} \right) \tilde{P}^{[ijk]}(\gamma_i), \end{aligned} \quad (5.14)$$

$$a_{2i}(\rho^2, \gamma) = -\frac{\rho_i^6}{96} \tilde{P}^{[i]''''}(\gamma_i) - \frac{\rho_i^4}{8} \tilde{P}^{[i]'}(\gamma_i) - \sum_{j \neq i}^n \frac{\rho_i^4 \rho_j^2}{16 \gamma_{ij}^2} \tilde{P}^{[ij]}(\gamma_i) + \sum_{j \neq i}^n \frac{\rho_i^4 \rho_j^2}{16 \gamma_{ij}} \tilde{P}^{[ij]'}(\gamma_i),$$

$$a_{3i}(\rho^2, \gamma) = -\frac{\rho_i^6}{32} \tilde{P}^{[i]''}(\gamma_i) - \frac{\rho_i^4}{8} \tilde{P}^{[i]}(\gamma_i) + \sum_{j \neq i}^n \frac{\rho_i^4 \rho_j^2}{16 \gamma_{ij}} \tilde{P}^{[ij]}(\gamma_i),$$

$$a_{4i}(\rho^2, \gamma) = -\frac{\rho_i^6}{16} \tilde{P}^{[i]'}(\gamma_i),$$

$$a_{5i}(\rho^2, \gamma) = -\frac{\rho_i^6}{16} \tilde{P}^{[i]}(\gamma_i).$$

The polynomial part $V'(x)$ was determined by the fact that y given by (5.13) may be written in the form (2.4) as well.

Step 5

Finally we use the inversion formulae for ρ_i^2 and γ_i , given by (5.3) and (5.6) respectively, to find y in the form to order S^3

$$y = V'(x) + \sum_{\nu=1}^5 \sum_{i=1}^n \frac{b_{\nu i}(S)}{(x - \alpha_i)^\nu} + O(S^4), \quad (5.15)$$

with

$$\begin{aligned} b_{1i}(S) = & -2S_i, \\ b_{2i}(S) = & \sum_{j \neq i}^n \left[\frac{2S_i^2 - 4S_i S_j}{\Delta_i(\alpha) \alpha_{ij}} + 4 \frac{S_i^3 - 6S_i^2 S_j + 3S_i S_j^2}{\Delta_i(\alpha)^2 \alpha_{ij}^3} - 4 \frac{S_i^3 - 3S_i^2 S_j + 3S_i S_j^2}{\Delta_i(\alpha) \Delta_j(\alpha) \alpha_{ij}^3} \right] \\ & + \sum_{\substack{j \neq k \\ j, k \neq i}}^n \left[4 \frac{S_i^3 - 8S_i^2 S_j + S_i S_j^2 + 2S_i^2 S_k + 4S_i S_j S_k}{\Delta_i(\alpha)^2 \alpha_{ij}^2 \alpha_{ik}} - 4 \frac{S_i^3 - 4S_i^2 S_j + 2S_i^2 S_k}{\Delta_i(\alpha) \Delta_j(\alpha) \alpha_{ij}^2 \alpha_{ik}} \right] \end{aligned}$$

$$\begin{aligned}
& +4 \frac{3S_i S_j^2 - 4S_i^2 S_j + 4S_i^2 S_k - 4S_i S_j S_k}{\Delta_i(\alpha) \Delta_j(\alpha) \alpha_{ij}^2 \alpha_{jk}} - 4 \frac{S_i S_j^2 + S_i S_k^2 - 2S_i S_j S_k}{\Delta_i(\alpha) \Delta_j(\alpha) \alpha_{ij} \alpha_{jk}^2} \\
& + 8 \frac{S_i S_j^2 - 2S_i S_j S_k}{\Delta_i(\alpha) \Delta_k(\alpha) \alpha_{ij} \alpha_{jk}^2} - 4 \frac{S_i^2 S_j + S_i^2 S_k - 2S_i S_j S_k}{\Delta_i(\alpha) \Delta_j(\alpha) \alpha_{ij} \alpha_{ik} \alpha_{jk}} \Big] \\
& + \sum_{\substack{j \neq k \neq l \\ j, k, l \neq i}}^n \left[\frac{\frac{2}{3} S_i^3 + 10S_i^2 S_j - 4S_i^2 S_k - 10S_i^2 S_l + 4S_i S_k S_l}{\Delta_i(\alpha)^2 \alpha_{ij} \alpha_{ik} \alpha_{il}} \right. \\
& + \frac{2S_i^2 S_j - \frac{16}{3} S_i S_j^2 - 4S_i S_j S_k + 12S_i S_j S_l - 4S_i S_k S_l}{\Delta_i(\alpha) \Delta_j(\alpha) \alpha_{ij} \alpha_{jk} \alpha_{jl}} \\
& \left. - 4 \frac{S_i^2 S_k - S_i^2 S_l}{\Delta_i(\alpha) \Delta_k(\alpha) \alpha_{ij} \alpha_{ik} \alpha_{kl}} + 8 \frac{S_i S_j S_k - S_i S_j S_l}{\Delta_i(\alpha) \Delta_k(\alpha) \alpha_{ij} \alpha_{jk} \alpha_{kl}} \right] + O(S^4), \\
b_{3i}(S) &= -\frac{2S_i^2}{\Delta_i(\alpha)} - \sum_{j \neq i}^n \left[\frac{4S_i(S_i - S_j)(S_i - 2S_j)}{\Delta_i(\alpha)^2 \alpha_{ij}^2} - \frac{4S_i^2(S_i - 2S_j)}{\Delta_i(\alpha) \Delta_j(\alpha) \alpha_{ij}^2} \right] \\
& - \sum_{\substack{j \neq k \\ j, k \neq i}}^n \left[\frac{4S_i(S_i - S_j)(S_i - 2S_k)}{\Delta_i(\alpha)^2 \alpha_{ij} \alpha_{ik}} - \frac{4S_i^2(S_j - S_k)}{\Delta_i(\alpha) \Delta_j(\alpha) \alpha_{ij} \alpha_{jk}} \right] + O(S^4), \\
b_{4i}(S) &= \sum_{j \neq i}^n \frac{4S_i^2(2S_i - 3S_j)}{\Delta_i(\alpha)^2 \alpha_{ij}} + O(S^4), \\
b_{5i}(S) &= -\frac{4S_i^3}{\Delta_i(\alpha)^2} + O(S^4).
\end{aligned}$$

Here the coefficient $b_{1i}(S)$ is the exact result obtained by comparing the term tending to $\frac{1}{x}$ at $x = \infty$ in (2.4) and (5.13). We would like to remark that the inversion formula (5.3) for ρ_i^2 to order S^2 is sufficient to evaluate the other coefficients $b_{\nu i}(S)$'s to order S^3 . Having explicitly known y in the expansion form (5.15) it is easy to evaluate the disc amplitude (2.8) for the n -cut solution. We then see that for $n = 2$ it exactly reproduces the disc amplitude given by (4.18). Moreover we note that the evaluation of (2.8) to order S^2 gives the disc amplitude (4.2) in the extended form to the generic n , which was obtained by the $\frac{1}{N}$ expansion in [8].

6. Conclusions

To the author's knowledge the disc amplitude for the multi-cut solution was calculated only the diagrammatic expansion in the literature. The result was merely reported to order S^2 for the generic n -cut solution [8]. The calculation to higher orders can be hardly done, because the diagrammatic expansion becomes complicate as the degree of the polynomial scalar potential increases. To confront with this situation we exploited the alternative method, *i.e.*, the complex analysis of the (hyper)elliptic curve. In section 4 the method was worked out to order S^3 for the disc amplitude

for the 2-cut solution. Dealing with the elliptic curve it is relatively easy to extend the calculation to higher orders. In section 5 the argument was generalized for the generic n -cut solution. By using a more systematic algorithm than in section 4, the disc amplitude for the n -cut solution was calculated up to order S^3 . There we have given the inversion formula for γ_i to order ρ^6 . We would like remark that it was sufficient to give the formula to order ρ^4 for the calculation of the planar amplitude to order S^3 [11]. It was checked that the disc amplitude thus obtained indeed reproduces the one for the 2-cut solution obtained by the $\frac{1}{N}$ expansion in [8]. The algorithm in this paper allows us to generalize the calculation of the disc amplitude for the n -cut solution to higher orders.

Note Added:

After posting the paper on the arXiv, the author was told that the disc amplitude for the n -cut solution was calculated to order S^3 by a different method in [13]. But the result (4.16) in [13] is erroneous. After the correction it reads

$$\begin{aligned} \mathcal{F}_3^{\chi=1} = & \sum_{f=1}^{N_f} \sum_{i=1}^n \left[-\frac{1}{2} \frac{S_i^3}{R_i^2 e_{if}^4} - \sum_{j \neq i}^n \left\{ \frac{2}{3} \frac{S_i^2 (2S_i - 3S_j)}{R_i^2 e_{ij} e_{if}^3} + \frac{2S_i^3 - 5S_i^2 S_j + 2S_i S_j^2}{R_i^2 e_{ij}^2 e_{if}^2} \right. \right. \\ & \left. \left. + \frac{6S_i^3 - 26S_i^2 S_j + 14S_i S_j^2}{R_i^2 e_{ij}^3 e_{if}} + \frac{2S_i^3 - 8S_i^2 S_j + 2S_i S_j^2}{R_i R_j e_{ij}^3 e_{if}} \right\} + \sum_{j \neq i}^n \sum_{k \neq j, i}^n \{ \cdots \} \right], \end{aligned}$$

by extracting the single sum contribution $\sum_{j \neq i} \{ \cdots \}$ from $\sum_{j \neq i} \sum_{k \neq i}$ and $\sum_{j \neq i} \sum_{k \neq j}$. Then this agrees with the disc amplitude given by (2.8) with (5.15) in our paper, up to the double sum contribution $\sum_{i=1} \sum_{j \neq i} \{ \cdots \}$. To check this, note that with the notation in [13]

$$\sum_{j \neq i}^n \frac{S_j^{N-1}}{e_{ij}^M R_j} = - \sum_{j \neq i}^n \frac{S_j^{N-1}}{e_{ij}^M R_i} + \sum_{j \neq i}^n \sum_{k \neq j, i}^n \{ \cdots \}, \quad M, N = 1, 2, \dots,$$

which follows from

$$\sum_{i=1}^n \oint_{e_i} \frac{dx}{(x - e_i)^M W'(x)} = 0, \quad \sum_{l=1}^n \sum_{i=1}^n \oint_{e_i} \frac{S_l^{N-1} dx}{(x - e_l)(x - e_i)^M W'(x)} = 0.$$

It is more involved to check the consistency between both results for the triple sum contribution $\sum_{i=1} \sum_{j \neq i} \sum_{k \neq j, i} \{ \cdots \}$. The above calculation uses the resolvent and the algorithm developed in [4, 13]. It is simple, but allows us to evaluate merely counter integrals of the DV differential. On the other hand the calculation in our paper uses the other algorithm [11]. It allows us to argue on the DV differential itself, expressing the branch points of the Riemann surface as functions of S_i 's. The latter algorithm is important to study the Whitham hierarchy of the DV theory.

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